フラクタル的乱流場の速度と渦度の分布(ナヴィエ・ストークス方程式の解と場の構造)

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フラクタル的乱流場の速度と渦度の分布

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" $f^{-\beta}$ Power Spectrum and Stable Distribution"

Abstract

Statistical properties of fractal noises with $f^{-\beta}$ power spectrum are analysed theoretically. Such noises are produced by fractional integral of white noises and their distribution can be calculated by applying Holtsmark's method. The obtained distributions are Levy's stable distributions and any stable distribution is derived from those noises. An application to turbulence is discussed.

Fluctuations with $f^{-\beta}$ type power spectrum can be found in many fields of science, and they are attracting much attention recently from the stand point of the fractal[1]. Kolmogorov's 5/3 power law in turbulence and $1/f$ noise in electric systems are good examples of these fluctuations. Although experimental discovery of these fluctuations was done several decades ago, theoretical analyses have not developed sufficiently.

Fractional Brownian motion[1] introduced by Mandelbrot is one of the most basic model for such fluctuations. It is defined by fractional integral of Gaussian white noise and has been applied, for example, to computer graphics for earth's relief[1]. However, little application has done in physics so far and much of its statistical properties are not yet elucidated.

In this letter, we are going to analyse generalized fractional Brownian motions with $f^{-\beta}$ power spectrum and show that the distribution of such fluctuations follow the stable law[2], namely, their distribution become Levy's stable distribution. Characteristic exponent of the stable distribution will be represented by a simple function of β, the exponent of the power spectrum. If we generalize the white noise to non-Gaussian, then we can obtain not only symmetric stable distributions but also any asymmetric ones. This may be the first report in which a total parameter family of stable distributions is derived from a physical model.

Let us consider a fluctuation $x(t)$ which is produced by a linear transformation of white noise $w(t)$ with a Green function $G(t)$:

$$x(t) = \int_{-\infty}^{\infty} G(t-t')w(t')dt' \quad . \qquad (1)$$

Power spectrum of $x(t)$, $S_x(f)$, is calculated by using the properties of white noise as,

$$S_x(f) = \frac{<w^2>}{2\pi} |\hat{G}(f)|^2 \quad , \qquad (2)$$

where $<w^2>$ is a constant which denotes the variance of the white noise and $\hat{G}(f)$ is the Fourier transform of $G(t)$. In order to make the power law of this spectrum, that is,

$$S_x(f) \propto f^{-\beta} \quad , \qquad (3)$$

one of the best candidates for the Green function is

$$G(t) = \begin{cases} 0 & t \leqq 0 \\ \dfrac{1}{\Gamma(\lambda)} t^{\lambda-1} & t > 0 \end{cases} \quad . \qquad (4)$$

Here $\lambda = \beta / 2$ and $\Gamma(\lambda)$ is the gamma function. This is the Green function for the fractional integral of order λ, which is well-known for mathematicians as Riemann-Liouville integral [3].

The distribution of the fluctuation $x(t)$, $P_t(x)$, can analytically be obtained in the following way by generalizing Holtzmark's method [4]. In order to calculate the distribution, we discretize Eq.(1) by considering the case that the white noise is composed of N sharp pulses in a finite range of t:

$$x(t)=\sum_{j=1}^{N} w_j\, G(t-t_j) \tag{5}$$

where $t_j \in [-T, T]$ and w_j denote the location and the magnitude of the j-th pulse, respectively. Then, we have the distribution as

$$\begin{aligned} P_t(x) &= \int\cdots\int \prod_{j=1}^{N} dw_j\, dt_j\, \delta\Big(x-\sum_{j=1}^{N} w_j G(t-t_j)\Big) \frac{1}{(2T)^N} \prod_{j=1}^{N} p(w_j) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho\ e^{-i\rho x}\ \Phi(\rho) \quad , \end{aligned} \tag{6}$$

where $p(w_j)$ is the distribution of w_j and

$$\Phi(\rho) = \left[\int_{-\infty}^{\infty} dw \int_{-T}^{T} dt' \frac{p(w)}{2T}\, e^{i\rho w G(t-t')}\right]^N \tag{7}$$

is the characteristic function for x. We now let N and T tend to infinity keeping $m = N/T$ constant. We thus obtain

$$\Phi(\rho) = e^{Z(\rho)} \quad , \tag{8}$$

where

$$Z(\rho) = m \int_{-\infty}^{\infty} dw\, p(w) \int_{-\infty}^{\infty} dt' \left\{ e^{i\rho w G(t-t')} - 1 \right\} \tag{9}$$

is the cumulant. The integral of t' in Eq.(9) can be carried out by introducing $s = |w/\Gamma(\lambda)|\,|t-t'|^{\lambda-1}$ as the variable of integration instead of t'. Then t dependence in the integral vanishes and Eq.(9) is transformed into

$$Z(\rho) = q\int_0^{\infty} ds\, s^{-1-\alpha}(\cos\rho s - 1) + ir\int_0^{\infty} ds\, s^{-1-\alpha}\sin\rho s \quad , \tag{10}$$

where

$$\alpha = \frac{1}{1-\lambda} = \frac{2}{2-\beta}$$
$$q = c\int_0^{\infty} dw\left\{p(w)+p(-w)\right\}w^{\alpha} \;,\; r = c\int_0^{\infty} dw\left\{p(w)-p(-w)\right\}w^{\alpha}$$
$$\text{and} \qquad c = \frac{m\,\alpha}{|\Gamma(\lambda)|^{\alpha}} \quad . \tag{11}$$

This cumulant, Eq.(10), directly leads the normal form of the stable distribution [2], if we adjust the mean and scale factor appropriately. We finally reduce the distribution of x (t), $P_t(x)$, to the following well-known normal form of the stationary stable distribution:

For $0 < \alpha < 1$ and $1 < \alpha \leqq 2$

$$P(x;\alpha,\gamma) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} d\rho \exp(-ix\rho - \rho^{\alpha} e^{i\pi\gamma/2}) \quad , \tag{12}$$

and for $\alpha = 1$

$$P(x;1,\tfrac{r}{q}) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} d\rho \exp\left\{-ix\rho - \rho(1 - i\tfrac{2}{\pi}\tfrac{r}{q}\log\rho)\right\} \quad , \tag{13}$$

where

$$\gamma = \frac{2}{\pi}\tan^{-1}\left(-\frac{r}{q}\tan\frac{\pi\alpha}{2}\right) \quad . \tag{14}$$

This parameter γ and also r/q in Eq.(13) dominate the symmetry of the distribution of x and they range

$$|\gamma| \leqq \begin{cases} \alpha & 0<\alpha<1 \\ 2-\alpha & 1<\alpha<2 \end{cases} \quad \text{and} \quad |\frac{r}{q}| \leqq 1 \quad . \tag{15}$$

In the case these parameters are zeros, the distributions are symmetric. For example, if the white noise is statistically symmetric, namely, $p(w)=p(-w)$, then γ and r/q vanish automatically, and the distribution of x becomes also symmetric as is expected directly from Eq.(1). In other cases such that $r \neq 0$, the distribution of x is asymmetric. Especially, stable distributions with extreme asymmetry are obtained when the distribution of the white noise is one-sided.

The characteristic exponent of the stable distribution, α, is always restricted within the above range, $0<\alpha \leqq 2$. This condition is equivalent to the restriction for β, $1 \geqq \beta$, which insure the stationarity of the random process $x(t)$. For $\beta > 1$ the process is not stationary and the distribution of $x(t)$ does not converge. $1/f$ power spectrum is obtained at the critical case $\beta = 1$, when $\alpha = 2$ and the distribution becomes Gaussian with diverging variance.

It is wellknown that every stable distribution except Gaussian $(\alpha \neq 2)$ possesses long-tail, that is, for sufficiently large $|x|$,

$$P(x;\alpha,\gamma) \propto |x|^{-\alpha-1} \quad . \tag{16}$$

This singular behaviour of x originates in the singularity of the Green function, $G(t)$ at $t=0$. Actually, we can show that the long-tail disappears if we introduce a cut-off t_c and modify the Green function such that $G(t) = \text{const.}$ for $0 < t \leqq t_c$ in Eq.(4).

In Eq.(4) we have applied Riemann-Liouville's Green function for the fractional integral, however, it is not the only choice. For example, Mandelbrot has introduced an anti-symmetric Green function [1]

$$G(t) = \frac{t}{\Gamma(\lambda)} |t|^{-\lambda} \quad . \tag{17}$$

In such case, the distribution of x is also stable with the same characteristic exponent as in Eqs.(12) and (13), but the symmetry parameters, γ and r/q, vanish automatically and we can obtain only symmetric stable distributions.

So far, we have considered that the coordinate t is in 1-dimensional Euclidian space. It is not difficult to generalize the space dimension to d, namely, we can consider the case that t is a vector with d-components in Eq.(1). The Green function for fractional integral of order λ is then generalized, for example, to

$$G(t) \propto |t|^{\lambda - d} \quad . \tag{18}$$

Further, we can generalize the spatial distribution of the white noise to fractal, that is the case that the distribution of the location of white noise (t_j in Eq.(5)) is not uniform but D dimensional. Omitting the calculation, we obtain the most generalized results: The power spectrum of $x(t)$ becomes

$$S_x(f) \propto f^{3d-2D-2\lambda-1} \quad . \tag{19}$$

The distribution of $x(t)$ becomes also the stationary stable distribution and its characteristic exponent is represented as

$$\alpha = \frac{D}{d-\lambda} \quad . \tag{20}$$

The symmetry parameters of the stable distribution, γ and r/q are almost identical to Eq.(14) and those in Eq.(11).

Applicability of these results are very wide. For example, let us first consider the Holtsmark's problem, that is, what is the distribution of the force acting on a star, per unit mass, due to the gravitational attraction of the neighboring stars? In this case, we regard t, x and $w(t')$ as the 3-dimensional coordinate, the gravitational force and the mass of star at t'. Then the spatial dimension $d=3$ and the inverse second power law is reduced to $\lambda=1$, hence from Eq.(20) we readily know that the desired distribution is the stable distribution with the characteristic exponent $\alpha=D/2$, where D denotes the fractal dimension of the stars. Here the Green function should be an odd function of t like Eq.(17), the obtained stable distribution is symmetric. In the special case that stars distribute uniformly throughout the universe, then $D=3$ and the distribution becomes the Holtsmark distribution [4]. (Remember that the Holtsmark distribution is the symmetric stable distribution with the charactristic exponent 3/2, $P(x\,;3/2,\,0)$ in Eq.(12).) This kind of generalization of the Holtsmark distribution has already been done by the author [5] in a more rigorous way.

More interesting applications are expected to be developed in the theory of turbulence. We can easily constitute a random velocity field with $k^{-\beta}$ power spectrum by the fractional integral of white noise. For example, Eq.(19) indicates that $k^{-5/3}$ power spectrum is obtained by the fractional integral of order 11/6 in 3-dimensional space. Mandelbrot has already proposed such model of turbulence about ten years ago [6]. The distribution of velocity of such random field can not be defined because the random field is not stationary. However, if we consider vorticity, instead of velocity, we can obtain its distribution by applying the above results. For vorticity is the first order derivative of velocity field, the order of fractional integration for the vorticity field, λ, becomes 5/6. Then from Eq.(20), we may conclude that the distribution of the vorticity is the stationary stable distribution with the characteristic exponent $\alpha = 18/13$. This indicates that the distribution of vorticity in real turbulence may have a long-tail as shown in Eq.(16).

It may be better to consider the effects of intermittency, finiteness of Reynolds number, the boundary condition, etc.. However, this letter is a preliminary report and detailed discussions about turbulence will be published elsewhere.

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